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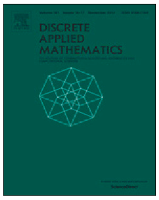

A geometric point of view on the synchronization of three Christoffel words

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A B S T R A C T

Let $G = (g_1, \dots, g_\ell)$ be a vector of positive integers and set $n = \sum_{i=1}^{\ell} g_i$. Writing $\mathrm{Ch}(a, b)$ for the Christoffel word with parameters (a, b), we study the following synchronization problem: choose one conjugate of each word $\mathrm{Ch}(g_i, n - g_i)$ so that, at every position $0, \dots, n-1$, exactly one of the chosen words contains the letter 1. This gives a gap-free form of superimposition, motivated by Fraenkel's conjecture.

We encode the choice of conjugates by a shift vector $V = (v_1, \dots, v_\ell)$. On the cyclic Cayley graph of $\mathbb{Z}/n\mathbb{Z}$, this yields an orbital matrix $O(G, V)$, with entries

$$o_{i,j} = (v_i + jg_i) \bmod n,$$

and a Christoffel–conjugate matrix $C(G, V)$, which records the columns where a wraparound occurs. The vector V is a synchronizing seed precisely when each column of $C(G, V)$ contains exactly one nonzero entry. The main algebraic tool is a vertical invariant: the column sums of $O(G, V)$ are constant if and only if this one-wraparound-per-column condition holds.

Using this criterion, we give explicit synchronizing seeds for every pair of Christoffel words of common length, for triples in which two generators coincide, for triples with three equal generators, and for the all-distinct family proportional to $(1, 2, 4)$. Finally, we give a geometric interpretation: each synchronized row is the Freeman chain code of a standard 4-connected Réveillès segment, whose parameter μ_i is the unique integer in $\{1 - n, \dots, 0\}$ satisfying $\mu_i \equiv -v_i \pmod{n}$.

1. Introduction

Motivation—Fraenkel's conjecture. Within Combinatorics on Words, a finite or infinite binary word is *balanced* if any two factors of the same length have numbers of 1's that differ by at most one. A long-standing conjecture of Fraenkel links balance and distinct letter frequencies on a k-letter alphabet: up to letter permutation, there should be a unique periodic balanced word whose k letter frequencies are

$$\frac{1}{2^k - 1}, \frac{2}{2^k - 1}, \dots, \frac{2^{k-1}}{2^k - 1}.$$

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Using properties of Beatty sequences, the conjecture has been proved for $k = 3, \dots, 7$ (see, e.g., [1,7,11,13]). Balanced words over $k > 2$ letters have been extensively studied; constructions via Sturmian sequences go back to Tijdeman and to Hubert (recasting Graham's work in the vocabulary of words) [4,5,12,14]. See also [9] for the episturmian setting.

Christoffel words and superimposition. Christoffel words are finite balanced words on the binary alphabet [2,3]. Paquin and Reutenauer [8] studied the *superimposition* of two Christoffel words, allowing a blank symbol, motivated by Fraenkel's conjecture. In contrast, we work with *synchronization*: given ℓ Christoffel words of length n, we seek conjugates so that, at each position, *exactly one* of the words has the letter 1 and no gap occurs.

From words to Cayley cycles (algebraic framework). We denote by $\mathbb{N}$ the set of whole numbers and by $\mathbb{N}^*$ the set of positive integers. Let $n \in \mathbb{N}^*$ and work in the Cayley cycle $\mathrm{Cay}(\mathbb{Z}/n\mathbb{Z}, \{g\})$. The word $\mathrm{Ch}(g, n-g)$ can be read by labelling edges $j \to (j+g) \bmod n$ as 0 or 1 depending on whether $j+g$ wraps around modulo n. For a generator vector $G = (g_1, \dots, g_\ell)$ and a seed $V = (v_1, \dots, v_\ell)$, define the *orbital* entries $o_{i,j} = (v_i + jg_i) \bmod n$ and mark a *decrease* when, for a given i and j, we have $o_{i,j+1} < o_{i,j}$. A seed *synchronizes* if each column has exactly one decrease; concatenating the row indices of these decreases yields the synchronized word over $\{1, \dots, \ell\}$.

Contributions. Our main contributions are:

- an algebraic framework via the matrices $O(G, V)$ and $C(G, V)$ and a one-line *vertical invariant* characterizing synchronization;
- a complete description of synchronizing seeds for $\ell = 2$; for $\ell = 3$, a general solution when at least two generators coincide, and a specific all-distinct family proportional to $(1, 2, 4)$;
- a geometric interpretation: once synchronized, each row is the Freeman code of a standard 4-connected Réveillès segment whose parameter is read directly from V.

Organization of the paper. Section 2 fixes notation and recalls the algebraic reading of Christoffel words via Cayley cycles and conjugates. Section 3 defines the two new matrices used for the synchronization. Section 4 gives arithmetic descriptions of the positions of the nonzero letter in conjugates. Section 5 states and proves the vertical invariant and collects basic consequences. Section 6 and Section 7 give the seed characterizations for $\ell = 2$ and $\ell = 3$, respectively. A *geometric* interpretation via Réveillès discrete segments is presented in Section 8. We conclude in Section 9.

Scope. Parts of the $\ell = 2$ case are classical. Our aim is a unified matrix/seed viewpoint with streamlined proofs, together with an explicit treatment of $\ell = 3$ in the cases above. We view this formulation as a step towards a fuller understanding of the synchronization problems motivated by Fraenkel's conjecture.

2. Christoffel words via Cayley cycles

We write $a \equiv b \pmod{n}$ when n divides $a - b$. An *alphabet* A is a finite set of symbols; its elements are *letters*. A *word* over A is a finite sequence of letters; the set of all words is A^*. The empty word is ε. For $w \in A^*$, the length of w is $|w|$, and $|w|_a$ denotes the number of occurrences of the letter $a \in A$ in w, so that $|w| = \sum_{a \in A} |w|_a$.

Concatenation on A^* is written by juxtaposition: for $u, v \in A^*$, uv denotes their concatenation. For $w \in A^*$ and $k \in \mathbb{N}$, set $w^0 = \varepsilon$ and $w^{k+1} = w\, w^k$ (i.e., w^k is w repeated k times). If $w = pfs$ for some $p, f, s \in A^*$, then f is a *factor* of w.

Two words $w, w' \in A^*$ are *conjugate* if there exist $u, v \in A^*$ with $w = uv$ and $w' = vu$. We denote by $w' \equiv_k w$ the conjugate obtained by cutting w after k letters, and by $\mathscr{L}(w)$ the set of all conjugates of w. For integers $a, b > 0$, we write $a \perp b$ when $\gcd(a, b) = 1$.

Throughout the paper, all Christoffel words are lower Christoffel words. For $a, b \in \mathbb{N}$, with $(a, b) \neq (0, 0)$, we denote by $\mathrm{Ch}(a, b)$ the Christoffel word with parameters (a, b), that is, the balanced word containing a letters 1 and b letters 0. When $\gcd(a, b) = 1$, we call $\mathrm{Ch}(a, b)$ the primitive Christoffel word of slope a/b. If $d = \gcd(a, b)$, then

$$\mathrm{Ch}(a, b) = \mathrm{Ch}\left(\frac{a}{d}, \frac{b}{d}\right)^d .$$

Thus slope terminology is used only for primitive Christoffel words, whereas possibly non-primitive Christoffel words are specified by their parameters. In particular, the same rational slope can determine different parameter pairs, and the pair (a, b) determines the possibly non-primitive word $\mathrm{Ch}(a, b)$.

Cayley-cycle definition of christoffel words. Let $a, b \in \mathbb{N}^*$ with $a \perp b$ and set $n = a + b$. Consider the Cayley graph

$$\mathrm{Cay}\big(\mathbb{Z}/n\mathbb{Z}, \{a\}\big),$$

whose vertex set is $\{0, 1, \dots, n-1\}$ and whose edges are $j \to (j + a) \bmod n$. For $j = 0, 1, \dots, n-1$, write each vertex as its representative in $\{0, \dots, n-1\}$ and label the edge $j \to (j + a) \bmod n$ by

$$\begin{cases} 1 & \text{if } j + a \geq n \quad (\text{a } \textit{decrease}\text{: it wraps modulo } n), \\ 0 & \text{otherwise.} \end{cases}$$

Reading these n labels in the order $0 \to a \to 2a \to \cdots \to (n-1)a$ (all modulo n) yields the primitive Christoffel word $\mathrm{Ch}(a, b)$ of slope a/b over the alphabet $\{0, 1\}$. It contains exactly a letters 1 and b letters 0.

We denote by $D_c(a, b) \subseteq \{0, \ldots, n-1\}$ the set of *decreasing positions*, i.e., the positions of the letter 1 in this word.

Theorem 1 (*Decreasing Positions; [8]*). *Let $a \perp b$ and $n = a + b$. Choose $\alpha \in \mathbb{Z}$ with $\alpha a \equiv -1 \pmod{n}$. Then*

$$D_c(a, b) = \left\{ (i\alpha) \bmod n \ : \ i = 1, \ldots, a \right\}.$$

Example 1. For $(a, b) = (3, 5)$, we have $n = 8$ and may take $\alpha \equiv 5 \pmod{8}$ (since $3 \cdot 5 \equiv -1 \pmod{8}$). Thus

$$D_c(3, 5) = \{\, 5,\ 2,\ 7 \,\} = \{2, 5, 7\},$$

so the primitive Christoffel word of slope $3/5$ is $w = 00100101$.

We remark the following: here $\alpha \equiv 5 \pmod{8}$ coincides numerically with b, but this is accidental and not true in general.

3. Synchronization of ℓ christoffel words

We now formalize synchronization for a family of Christoffel words of common length. Throughout, let

$$G = (g_1, \ldots, g_\ell) \in (\mathbb{N}^*)^\ell, \qquad n = \sum_{i=1}^{\ell} g_i,$$

and recall that the word attached to g_i is $\mathrm{Ch}(g_i, n - g_i)$, which contains exactly g_i letters 1 and $n - g_i$ letters 0. Whenever elements of $\mathbb{Z}/n\mathbb{Z}$ are compared by $<$, we use their representatives in $\{0, \ldots, n-1\}$.

Definition 1 (*Orbital and Christoffel–Conjugate Matrices*). Let $V = (v_1, \ldots, v_\ell) \in \mathbb{N}^\ell$. The *orbital matrix* $O(G, V) = (o_{i,j})$ has rows indexed by $i = 1, \ldots, \ell$ and columns by $j = 0, \ldots, n$, and is defined by

$$o_{i,j} = (v_i + j\, g_i) \bmod n,$$

The *Christoffel–conjugate matrix* $C(G, V) = (c_{i,j})$ has rows $i = 1, \ldots, \ell$ and columns $j = 0, \ldots, n-1$, and is defined by

$$c_{i,j} = \begin{cases} i, & \text{if } o_{i,j+1} < o_{i,j} \quad \text{(a wrap, i.e. a \textit{decrease})}, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 1 (*Meaning of the Rows*). For fixed i, the row $(o_{i,j})_{j=0}^{n}$ lists the Cayley orbit $v_i,\ v_i + g_i, \ldots, v_i + n g_i$ modulo n. The decreases in that row occur exactly at the positions of the letter 1 in the conjugate of $\mathrm{Ch}(g_i, n - g_i)$ whose orbit starts at v_i (cf. Theorem 1). Thus, each row of $C(G, V)$ marks the 1-positions of that conjugate by the row index i.

Definition 2 (*Seed and Synchronization*). We say that V *synchronizes* G (or that V is a *seed* for G) if *every* column of $C(G, V)$ contains exactly one nonzero entry (equivalently: exactly one decrease occurs between columns j and $j{+}1$ for each $j = 0, \ldots, n-1$).

Definition 3 (*Synchronized Word*). When V is a seed, the *synchronized word* is the length-n word over the alphabet $\{1, \ldots, \ell\}$ obtained by reading, column by column, the unique nonzero entry of $C(G, V)$. In particular, it has no 0s. This is a gap-free strengthening of *superimposition* in the sense of [8].

Remark 2 (*Shifting the Seed*). If V is a seed, then for any $t \in \{0, \ldots, n-1\}$ the column vector

$$V^{(t)} = (\, o_{1,t}, \ldots, o_{\ell,t} \,)$$

is also a seed for G. Intuitively, advancing all orbits by one column cyclically permutes the columns of $C(G, V)$ and preserves the one-decrease-per-column property.

Example 2. Let $G = (5, 3)$, so $n = 8$, and take $V_0 = (0, 0)$. Then

$$O(G, V_0) = \begin{pmatrix} 0 & 5 & 2 & 7 & 4 & 1 & 6 & 3 & 0 \\ 0 & 3 & 6 & 1 & 4 & 7 & 2 & 5 & 0 \end{pmatrix}, \quad C(G, V_0) = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & 0 & 2 & 0 & 2 \end{pmatrix}.$$

The last column shows a collision (both rows decrease at $j = 7$), so V_0 is not a seed. With $V = (0, 7)$ we obtain

$$O(G, V) = \begin{pmatrix} 0 & 5 & 2 & 7 & 4 & 1 & 6 & 3 & 0 \\ 7 & 2 & 5 & 0 & 3 & 6 & 1 & 4 & 7 \end{pmatrix}, \qquad C(G, V) = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 2 & 0 & 2 & 0 & 0 & 2 & 0 & 0 \end{pmatrix},$$

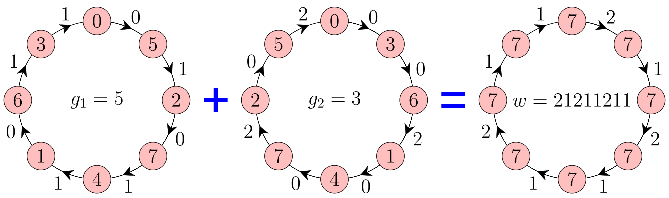


Fig. 1. The representation as Cayley graphs for each row of $C(G, V)$, for $G = (5, 3)$. The two cycles on the left of the equal sign show the situation for $V = (0, 0)$. The synchronization of both generators (right side of the equal sign), obtained by some cyclic permutations corresponding to $V = (0, 7)$, gives the synchronized word $w = 21211211$.

which has exactly one nonzero per column; hence V synchronizes G. The synchronized word is

$$w = 21211211.$$

See Fig. 1 for an illustration of that example in terms of Cayley graphs.

4. Positions of letter 1 for conjugates of christoffel words

Let $w = \mathrm{Ch}(a, b)$ be the primitive Christoffel word of slope $\frac{a}{b}$ with $a \perp b$, and put $n = a+b$. Its Cayley orbit $O(w)$ is the list of vertices $0, a, 2a, \ldots, (n-1)a$ in $\mathbb{Z}/n\mathbb{Z}$. If w' is a conjugate of w, then the orbit $O(w')$ starts at some label $p \in \mathbb{Z}/n\mathbb{Z}$ lying in the cyclic subgroup generated by a, equivalently

$$a\,x \equiv -p \pmod{n} \quad \text{for some } x \in \{0, \ldots, n-1\}.$$

We are interested in the set of positions of the letter 1 (the *decreasing positions*) in w and in any of its conjugates w'.

Definition 4. For the primitive Christoffel word $w = \mathrm{Ch}(a, b)$ of slope $\frac{a}{b}$, with $n = a + b$, and for a conjugate w' whose orbit starts at label $p \in \mathbb{Z}/n\mathbb{Z}$, define

$$P(a, b, p) \;=\; \{\, i \in \{0, \ldots, n-1\} : w'[i] = 1 \,\}.$$

Coprime case. Recall from Theorem 1 that if $\alpha a \equiv -1 \pmod{n}$ then

$$D_c(a, b) = \{\, (i\alpha) \bmod n : i = 1, \ldots, a \,\}$$

is the set of decreasing positions of w (i.e., the 1-positions of the conjugate whose orbit starts at $p = 0$). For a general starting label p we have:

Proposition 1. *Let $a \perp b$, $n = a + b$, and choose α with $\alpha a \equiv -1 \pmod{n}$. If w' starts at label $p \in \mathbb{Z}/n\mathbb{Z}$, pick $x \in \mathbb{Z}$ with*

$$a\,x \equiv -p \pmod{n}.$$

Then

$$P(a, b, p) \;=\; \{\, (i\alpha + x) \bmod n : i = 1, \ldots, a \,\}.$$

Proof. Starting the Cayley orbit at p instead of 0 amounts to translating the index set by x with $ax \equiv -p \pmod{n}$. Since $D_c(a, b) = \{(i\alpha) \bmod n\}$, we obtain $P(a, b, p) = (D_c(a, b) + x) \bmod n$. □

Example 3. Take $(a, b) = (4, 7)$ so $n = 11$. Choose $\alpha \equiv 8 \pmod{11}$ since $4 \cdot 8 \equiv -1 \pmod{11}$. Thus $D_c(4, 7) = \{\, 8, 5, 2, 10 \,\} = \{2, 5, 8, 10\}$. The primitive Christoffel word of slope 4/7 is $w = 00100100101$ and $O(w)$ starting at 0 gives:

0 − 4 − 8 ━ 1 − 5 − 9 ━ 2 − 6 − 10 ━ 3 − 7 ━ 0

In the list here and later, we use as convention that the positions of the 1 are represented by a thick dash, printed in green in the electronic version.

If the orbit starts at $p = 3$, solve $4x \equiv -3 \equiv 8 \pmod{11}$, giving $x \equiv 2$. Then

$$P(4, 7, 3) = \{\, (i \cdot 8 + 2) \bmod 11 : i = 1, \ldots, 4 \,\} = \{10, 7, 4, 1\}.$$

3 − 7 ━ 0 − 4 − 8 ━ 1 − 5 − 9 ━ 2 − 6 − 10 ━ 3

Non-coprime case. When $d = \gcd(a, b) > 1$, the word $\mathrm{Ch}(a, b)$ is the dth power of the primitive Christoffel word $\mathrm{Ch}(a/d, b/d)$. The 1-positions form a union of d arithmetic progressions spaced by n/d:

Lemma 1. *Let $n = a + b$ and $d = \gcd(a, b)$. Choose $\alpha \in \mathbb{Z}$ with*

$$\alpha a \equiv -d \pmod{n}.$$

Assume that $p \equiv 0 \pmod{d}$, and let x be any solution of

$$a x \equiv -p \pmod{n}.$$

Then

$$P(a, b, p) = \bigcup_{j=0}^{d-1} \left\{ ((i\alpha + x) + j\,\tfrac{n}{d}) \bmod n : i = 1, \ldots, \tfrac{a}{d} \right\}.$$

Example 4 (*A Larger Non-Coprime Instance*). Let $(a, b) = (12, 15)$ so $n = 27$ and $d = 3$. Take α with $\alpha \cdot 12 \equiv -3 \equiv 24 \pmod{27}$; e.g. $\alpha \equiv 2 \pmod{9}$ (choose $\alpha = 2$). Let the orbit start at $p = 9$. Solve $12x \equiv -9 \equiv 18 \pmod{27}$, i.e. $4x \equiv 6 \pmod{9}$, giving $x \equiv 6 \pmod{9}$ (choose $x = 6$). Then, since $a/d = 4$ and $n/d = 9$,

$$\begin{aligned} P(12, 15, 9) &= \bigcup_{j=0}^{2} \{ ((2i + 6) + 9j) \bmod 27 : i = 1, 2, 3, 4 \} \\ &= \{8, 10, 12, 14\} \cup \{17, 19, 21, 23\} \cup \{26, 1, 3, 5\} \\ &= \{1, 3, 5, 8, 10, 12, 14, 17, 19, 21, 23, 26\}. \end{aligned}$$

Since

$$\mathrm{Ch}(12, 15) = \mathrm{Ch}(4, 5)^3$$

and $\mathrm{Ch}(4, 5) = 001010101$, we have $w = \mathrm{Ch}(12, 15) = (001010101)^3$. The orbit $O(w)$ starting at 0 gives:

$$0(-12 - 24 \text{ — } 9 - 21 \text{ — } 6 - 18 \text{ — } 3 - 15 \text{ — } 0)^3$$

Remark 3. In particular, for $p = 0$ the set $P(a, b, 0)$ reduces to $D_c(a, b)$ in the coprime case, and to a union of d evenly spaced progressions in the non-coprime case.

5. Vertical invariant

We work with the matrices $O(G, V)$ and $C(G, V)$ from Section 3, with

$$G = (g_1, \ldots, g_\ell) \in (\mathbb{N}^*)^\ell, \qquad V = (v_1, \ldots, v_\ell) \in \mathbb{N}^\ell, \qquad n = \sum_{k=1}^{\ell} g_k.$$

For $j = 0, \ldots, n$, define the *vertical sum*

$$A_j = \sum_{i=1}^{\ell} o_{i,j} \qquad \text{and} \qquad A = (A_j)_{j=0,\ldots,n}.$$

Theorem 2 (*Vertical Invariant*). *Let $d_j = \#\{ i : o_{i,j+1} < o_{i,j} \}$ be the number of* decreases *between columns j and $j+1$, for $j = 0, \ldots, n-1$. Then*

$$A_{j+1} - A_j = n\big(1 - d_j\big) \qquad (j = 0, \ldots, n-1).$$

In particular, A is constant if and only if exactly one decrease occurs between each pair of successive columns j and $j+1$.

Proof. In row i, from column j to $j+1$ we either add g_i (no wrap) or add $g_i - n$ (wrap/decrease). Summing over rows yields $A_{j+1} - A_j = \sum_i g_i - n\,d_j = n(1 - d_j)$. Hence $A_{j+1} = A_j$ if and only if $d_j = 1$ for all j. $\square$

When V is a seed (Definition 2), A is constant by Theorem 2. In this case, we call the common value

$$I(G, V) = A_0 = A_1 = \cdots = A_n$$

the *vertical invariant*.

Proposition 2 (*Value of the Invariant*).

Let $d_i = \gcd(g_i, n)$ *and let* r_i *be the least non-negative residue of* v_i *modulo* d_i*:*

$$r_i \;=\; v_i \bmod d_i, \qquad 0 \le r_i < d_i.$$

Then, for any G, V*,*

$$\frac{1}{n}\sum_{j=0}^{n-1} A_j \;=\; \frac{\ell n}{2} \;-\; \frac{1}{2}\sum_{i=1}^{\ell} d_i \;+\; \sum_{i=1}^{\ell} r_i.$$

In particular, if V *is a seed (so* A *is constant), then*

$$I(G,V) \;=\; \frac{\ell n}{2} \;-\; \frac{1}{2}\sum_{i=1}^{\ell} d_i \;+\; \sum_{i=1}^{\ell} r_i \qquad \textit{for each } j.$$

Proof. Since A is column-periodic modulo n, its arithmetic mean equals

$$\frac{1}{n}\sum_{j=0}^{n-1} A_j \;=\; \sum_{i=1}^{\ell} \frac{1}{n}\sum_{j=0}^{n-1} o_{i,j}.$$

Fix i. As j runs, $(o_{i,j})_{j=0}^{n-1}$ visits each element of the congruence class $\{\, r \in \{0,\dots,n-1\} : r \equiv v_i \pmod{d_i} \,\}$ exactly d_i times. Hence

$$\sum_{j=0}^{n-1} o_{i,j} \;=\; d_i \sum_{k=0}^{\frac{n}{d_i}-1} \bigl(r_i + k\,d_i\bigr) \;=\; n r_i + \frac{1}{2}\bigl(n^2 - n d_i\bigr).$$

Dividing by n and summing over i gives the claim. □

Corollary 1. *If one* chooses *an orbit that starts with* $v_i \equiv 0 \ (\mathrm{mod}\, d_i)$ *for all* i *(so* $r_i = 0$*), then* Proposition 2 *reduces to*

$$I(G,V) \;=\; \frac{\ell n}{2} \;-\; \frac{1}{2}\sum_{i=1}^{\ell} \gcd(g_i, n).$$

Example 5.

- $\ell = 2$, $G = (5,3)$, $n = 8$, $d_1 = d_2 = 1$. For the seed $V = (0,7)$ (which synchronizes), $r_1 = r_2 = 0$ and

  $$I(G,V) = \frac{2\cdot 8}{2} - \frac{1+1}{2} + 0 = 7,$$

  matching the constant column sum observed directly in Example 2.
- Nontrivial residues: $G = (4,6)$, $n = 10$, $d_1 = d_2 = 2$. For $V = (1,5)$ we have $r_1 = r_2 = 1$. Proposition 2 gives

  $$\frac{1}{10}\sum_{j=0}^{9} A_j = \frac{2\cdot 10}{2} - \frac{2+2}{2} + (1+1) = 10.$$

  (Here we only claim the average value; V need not be a seed.)

6. Seeds for two generators

Paquin and Reutenauer [8] proved that two Christoffel words of common length n can always be synchronized (in their terminology: superimposed, allowing a gap; here we require no gaps). In our framework, this becomes a simple, explicit seed.

Let $G = (g_1, g_2)$ with $n = g_1 + g_2$ and put $d = \gcd(g_1, g_2)$. Recall that for $\ell = 2$ we have $d_1 = \gcd(g_1, n) = d_2 = \gcd(g_2, n) = d$. From Proposition 2,

$$\frac{1}{n}\sum_{j=0}^{n-1}\sum_{i=1}^{2} o_{i,j} \;=\; n - \frac{d}{1} \;+\; (v_1 \bmod d) + (v_2 \bmod d).$$

In particular, if V is a seed (and hence the vertical sum is constant), then $I(G,V) = n - d + (v_1 \bmod d) + (v_2 \bmod d)$. Choosing residues $v_i \bmod d$ equal to 0 makes $I(G,V) = n - d$.

Proposition 3 (*Explicit Seed for $\ell = 2$*). *For every $G = (g_1, g_2)$ with $n = g_1 + g_2$ and $d = \gcd(g_1, g_2)$, the vector*

$$V = (0,\ n - d)$$

is a seed for G; hence $C(G, V)$ has exactly one nonzero per column and the synchronized word has length n with alphabet $\{1, 2\}$.

Proof (arithmetic partition). Work with representatives in $\{0, \dots, n-1\}$. In column j, a decrease occurs in row i if and only if

$$o_{i,j}\ \geq\ n - g_i.$$

With $V = (0, n-d)$, we have $o_{1,j} = (jg_1) \bmod n$ and $o_{2,j} = (n - d + jg_2) \bmod n$, thus

$$o_{1,j} + o_{2,j} \equiv n - d \pmod{n}, \qquad o_{1,j}, o_{2,j} \in d\mathbb{Z}.$$

Hence for each column j exactly one of the two inequalities

$$o_{1,j} \geq n - g_1 = g_2 \quad \text{or} \quad o_{2,j} \geq n - g_2 = g_1$$

holds: the forbidden gaps $[g_2 - d + 1, g_2 - 1]$ and $[g_1 - d + 1, g_1 - 1]$ contain no multiples of d. Therefore the wrap count per column is 1, and the claim follows from Theorem 2. □

Remark 4. For $\ell = 2$, $I(G, V) = n - d$ for any seed V. If $g_1 \perp g_2$ then $d = 1$ and $I(G, V) = n - 1$, independent of the seed V.

Example 6. Let $G = (2, 4)$ so $n = 6$ and $d = 2$. By Proposition 3, $V = (0, 4)$ synchronizes G. One computes

$$O(G,V) = \begin{pmatrix} 0 & 2 & 4 & 0 & 2 & 4 & 0 \\ 4 & 2 & 0 & 4 & 2 & 0 & 4 \end{pmatrix}, \qquad C(G,V) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 2 & 2 & 0 & 2 & 2 & 0 \end{pmatrix},$$

and the synchronized word is $w = 221221$, with $I(G, V) = 6 - 2 = 4$.

7. Seeds for three generators

We now treat $G = (g_1, g_2, g_3)$ with $n = g_1 + g_2 + g_3$. We give complete synchronization for the cases where two generators coincide (and for the degenerate case of three equal generators), and we state a uniform seed for the classical all-distinct family proportional to $(1, 2, 4)$.

7.1. Two equal generators

Assume $G = (g, g, g_3)$ with $g \perp g_3$ and $n = 2g + g_3$.

Proposition 4 (*GCDs and Parity*). *For $G = (g, g, g_3)$ with $g \perp g_3$ and $n = 2g + g_3$ we have*

$$\gcd(n, g) = 1, \qquad \gcd(n, g_3) = \gcd(2\,g, g_3).$$

Hence n is odd if and only if $\gcd(n, g_3) = 1$, and n is even if and only if $\gcd(n, g_3) = 2$.

Proof. Using the Euclidean algorithm, we have: $\gcd(n, g) = \gcd(2g + g_3, g) = \gcd(g_3, g) = 1$. Also $\gcd(n, g_3) = \gcd(2g + g_3, g_3) = \gcd(2\,g, g_3)$. Parity follows since $2g$ is even. □

Lemma 2 (*Value of the Invariant*). *With $G = (g, g, g_3)$ as above,*

$$I(G,V) = \begin{cases} \dfrac{3}{2}(n-1), & n \text{ odd}, \\ \dfrac{3n-4}{2}, & n \text{ even}, \end{cases}$$

for every seed V.

Proof. From Corollary 1 with $d_1 = d_2 = 1$ and $d_3 = \gcd(n, g_3) \in \{1, 2\}$. □

Theorem 3 (*Seeds When Two Generators Coincide*). *Let $G = (g, g, g_3)$ with $g \perp g_3$ and $n = 2g + g_3$. The vector*

$$V = \begin{cases} (\,n-1,\ \tfrac{n-1}{2},\ 0\,), & n \text{ odd}, \\ (\,n-1,\ \tfrac{n-2}{2},\ 0\,), & n \text{ even}, \end{cases}$$

synchronizes G.

Proof. Let $G = (g, g, g_3)$ with $n = 2g + g_3$ and $g \perp g_3$. By Proposition 1, the positions of the letter 1 in any conjugate are given by arithmetic progressions modulo n.

With the seed V given in the statement, let A and B be the sets of decreasing positions of the two conjugates of $\mathrm{Ch}(g, n - g)$ starting at v_1 and v_2, and let C be the set of decreasing positions of $\mathrm{Ch}(g_3, n - g_3)$ starting at 0.

Using Proposition 1 and the parity properties of n (Proposition 4), the three sets A, B, C are arithmetic progressions modulo n whose step sizes and offsets depend only on g, g_3, and the chosen seed. A direct modular analysis shows that these three sets are pairwise disjoint and that $|A| = |B| = g$ and $|C| = g_3$, hence

$$A \sqcup B \sqcup C = \{0, 1, \dots, n - 1\}.$$

Therefore exactly one decrease occurs in each column of $C(G, V)$, and V is a synchronizing seed for G. □

For a complete modular case analysis, see Appendix A.

Example 7. Let $G = (4, 4, 3)$ so $n = 11$ (odd). By Theorem 3, $V = (10, 5, 0)$ is a seed, and

$$O(G,V) = \begin{pmatrix} 10 & 3 & 7 & 0 & 4 & 8 & 1 & 5 & 9 & 2 & 6 & 10 \\ 5 & 9 & 2 & 6 & 10 & 3 & 7 & 0 & 4 & 8 & 1 & 5 \\ 0 & 3 & 6 & 9 & 1 & 4 & 7 & 10 & 2 & 5 & 8 & 0 \end{pmatrix}, \quad C(G,V) = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 3 \end{pmatrix}.$$

The synchronized word is $w = 12132123123$. By Lemma 2, $I(G, V) = \frac{3}{2}(11 - 1) = 15$.

Example 8. Let $G = (3, 3, 4)$ so $n = 10$ (even). Then $V = (9, 4, 0)$ is a seed and

$$O(G,V) = \begin{pmatrix} 9 & 2 & 5 & 8 & 1 & 4 & 7 & 0 & 3 & 6 & 9 \\ 4 & 7 & 0 & 3 & 6 & 9 & 2 & 5 & 8 & 1 & 4 \\ 0 & 4 & 8 & 2 & 6 & 0 & 4 & 8 & 2 & 6 & 0 \end{pmatrix}, \quad C(G,V) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 & 0 & 3 & 0 & 3 \end{pmatrix}.$$

Here $I(G, V) = \frac{3n-4}{2} = 13$.

Remark 5 (*Lifting by a Common Factor*)**.** If $\gcd(g, g_3) = d > 1$, set $g = dg'$, $g_3 = dg'_3$ and $n = dn'$. If $V' = (v'_1, v'_2, v'_3)$ is a seed for $G' = (g', g', g'_3)$ modulo n', then $V = dV' = (dv'_1, dv'_2, dv'_3)$ is a seed for G modulo n. At the matrix level, each row of $O(G', V')$ is lifted to $O(G, V)$ by multiplying entries by $d \pmod n$ and repeating the length-n' pattern d times; decreases repeat with period n', so the one-per-column property is preserved.

Example 9. Let $G = (1, 1, 2)$ with $n = 4$ and the seed vector $V = (3, 1, 0)$. We have $I(G, V) = 4$ and the orbital matrix is given by:

$$O(G,V) = \begin{pmatrix} 3 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 & 1 \\ 0 & 2 & 0 & 2 & 0 \end{pmatrix}.$$

For the vector $G' = (3, 3, 6)$, we can calculate immediately $V' = 3 \cdot V = (9, 3, 0)$, and both $O(G', V')$ and $C(G', V')$ can be deduced from $O(G, V)$ and $C(G, V)$:

$$O(G',V') = \begin{pmatrix} 9 & 0 & 3 & 6 & 9 & 0 & 3 & 6 & 9 & 0 & 3 & 6 & 9 \\ 3 & 6 & 9 & 0 & 3 & 6 & 9 & 0 & 3 & 6 & 9 & 0 & 3 \\ 0 & 6 & 0 & 6 & 0 & 6 & 0 & 6 & 0 & 6 & 0 & 6 & 0 \end{pmatrix},$$

$$C(G',V') = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 3 & 0 & 3 & 0 & 3 & 0 & 3 & 0 & 3 \end{pmatrix}.$$

The synchronized word is $w = (1323)^3$, where $I(G', V') = 12 = 3I(G, V)$.

7.2. Three equal generators

Theorem 4 (*Three Identical Generators*)**.** *Let $G = (g, g, g) = g(1, 1, 1)$ with $n = 3g$. Then $V = (2\,g, g, 0) = g(2, 1, 0)$ is a seed, and the synchronized word is $w = (321)^g$. Moreover $I(G, V) = 3g$.*

Proof. Factor out g and synchronize $G' = (1, 1, 1)$ with seed $V' = (2, 1, 0)$; lifting by g gives the claim (cf. Remark 5). The value of $I(G, V)$ follows either from the seed entries or from Proposition 2 with $d_i = \gcd(g, 3\,g) = g$. □

7.3. All distinct generators: the powers-of-two family

The classical Fraenkel family $(1, 2, 4)$ modulo 7 (and its multiples) admits a uniform seed in our framework; more generally:

Theorem 5 (*Uniform Seed for $G = (1, 2, \dots, 2^{\ell-1})$*)**.** *Let $\ell \geq 2$, put $G = (1, 2, \dots, 2^{\ell-1})$ and $n = 2^{\ell} - 1$. Then the vector*

$$V = \Big(\underbrace{\tfrac{n-1}{2}, \dots, \tfrac{n-1}{2}}_{\ell \text{ entries}}\Big)$$

is a seed for G. In particular $I(G, V) = \frac{\ell}{2}(n-1)$.

Proof. The proof follows the same reasoning as in the proof of Theorem 3, considering the following adjustments. Since n is odd, all $\gcd(2^k, n) = 1$, for all $k \in \{0, \dots, \ell - 1\}$. In column j, row k decreases if and only if $o_{k,j} \in [n - 2^k, n - 1]$. With the common start $v = (n-1)/2$, the maps $j \to o_{k,j}$ are permutations of $\{0, \dots, n-1\}$. One checks that the sets

$$S_k = \{\, j : \; o_{k,j} \in [n - 2^k, n - 1] \,\}$$

are pairwise disjoint and $|S_k| = 2^k$, so $\bigsqcup_k S_k = \{0, \dots, n-1\}$. (Equivalently: each residue $(j + v) \bmod n$ has a unique dyadic order in $\{1, \dots, n\}$, selecting exactly one k.) Thus the wrap count per column is 1. □

Example 10. For $G = (1, 2, 4)$ we have $n = 7$ and $V = (3, 3, 3)$. Then

$$O(G, V) = \begin{pmatrix} 3 & 4 & 5 & 6 & 0 & 1 & 2 & 3 \\ 3 & 5 & 0 & 2 & 4 & 6 & 1 & 3 \\ 3 & 0 & 4 & 1 & 5 & 2 & 6 & 3 \end{pmatrix}, \quad C(G, V) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 2 & 0 \\ 3 & 0 & 3 & 0 & 3 & 0 & 3 \end{pmatrix},$$

and the synchronized word is $w = 3231323$. Here $I(G, V) = \frac{3}{2}(7 - 1) = 9$.

Remark 6 (*Multiples of* $(1, 2, 4)$)**.** If $G' = d \cdot (1, 2, 4)$ and $n' = d \cdot 7$, then $V' = d \cdot (3, 3, 3)$ is a seed and $I(G', V') = d \cdot 9$ by the lifting principle in Remark 5.

8. Réveillès segments: a geometric interpretation

Réveillès lines and standard segments (freeman-first selection). We fix the binary alphabet $\{0, 1\}$, interpreted as *right* and *up* steps, respectively.

Definition 5 (*Freeman Chain Code*). A *north-east* lattice path is a finite sequence $P = (p_0, \dots, p_m)$ in $\mathbb{Z}^2$ with $p_{i+1} - p_i \in \{(1, 0), (0, 1)\}$ for all i (hence 4-connected with no diagonal moves). Its *Freeman chain code* is the word $F(P) \in \{0, 1\}^m$ defined by

$$F(P)[i] = \begin{cases} 0, & \text{if } p_{i+1} - p_i = (1, 0) \quad \text{(right step)}, \\ 1, & \text{if } p_{i+1} - p_i = (0, 1) \quad \text{(up step)}. \end{cases}$$

Let $a, b, \mu, \omega \in \mathbb{Z}$ with $a \perp b$ and $b > 0$. The (lower) Réveillès line of slope a/b, with intercept control μ, and thickness ω [10] is

$$D(a, b, \mu, \omega) \; = \; \{(x, y) \in \mathbb{Z}^2 : \; \mu \leq ax - by < \mu + \omega\}.$$

We work with the *standard* thickness $\omega = a + b$, abbreviated $D(a, b, \mu)$, and restrict to abscissas $x \in [0, b]$, and $\mu \in \{-a - b + 1, \dots, 0\}$. We later, in Remark 7, generalize for all $\mu \in \mathbb{Z}$.

Definition 6 (*Standard Finite Segment Selected Via Freeman*)**.** Set $n = a + b$. For $\mu \in \{1 - n, \dots, 0\}$, the *standard Réveillès segment* [10] $D[a, b, \mu]$ is the unique north-east path $P = (p_0, \dots, p_n) \subset D(a, b, \mu)$ with $x(p_0) = 0$, $x(p_n) = b$ and $|P| = n + 1$, whose first step is fixed by μ as follows:

$$F(P)[0] = \begin{cases} 0, & \text{if } \mu \in \{-b + 1, \dots, 0\}, \\ 1, & \text{if } \mu \in \{-a - b + 1, \dots, -b\}. \end{cases}$$

We write $\mathscr{R}(D[a, b, \mu]) = F(P) \in \{0, 1\}^n$ for its Freeman code (hence of length $a + b$).

Example 11. Following Definition 6, the Freeman chain codes of the discrete segments $D[3, 5, -6]$ and $D[3, 5, -2]$ are obtained by first determining the Reveillès discrete lines and then coding the segments as follows.

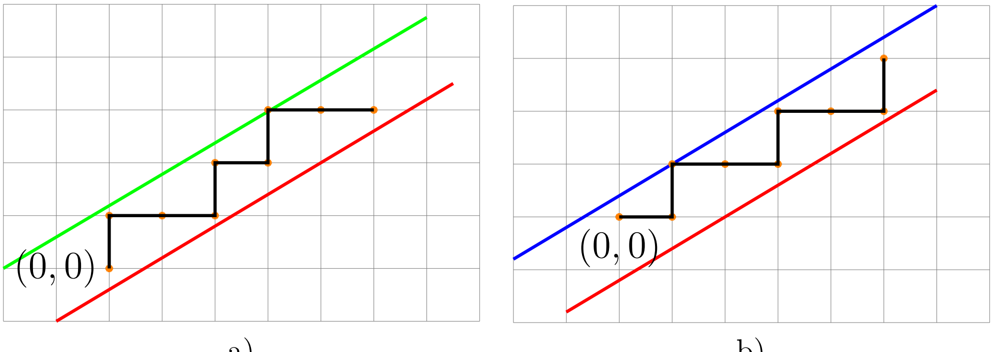
For $D[3, 5, -6]$, the Reveillès discrete line is formed from: $-6 \leq 3x - 5y < 2$, and the Freeman chain code is 10010100. For $D[3, 5, -2]$, we need: $-2 \leq 3x - 5y < 6$, and its coding gives: 01001001, as shown in Fig. 2.

In [6, Lemma 4.5], the Freeman chain code of the discrete line $D(a, b, 0)$ is the primitive Christoffel word of slope $\frac{a}{b}$ repeated infinitely many times. With Definition 6, $D[a, b, 0]$ represents the primitive Christoffel word of slope $\frac{a}{b}$.

Example 12. Let $(a, b) = (3, 5)$. The Reveillès standard segment $D[3, 5, 0]$ is obtained from the inequalities

$$0 \leq 3x - 5y < 8.$$

Since $\mu = 0$, the Freeman code of the path is the primitive Christoffel word of slope $\frac{3}{5}$, which is: 00100101, see Fig. 3.



a) b)

Fig. 2. Panel (a) represents the Freeman chain code for $D[3, 5, -6]$, and panel (b) represents the Freeman chain code for $D[3, 5, -2]$.

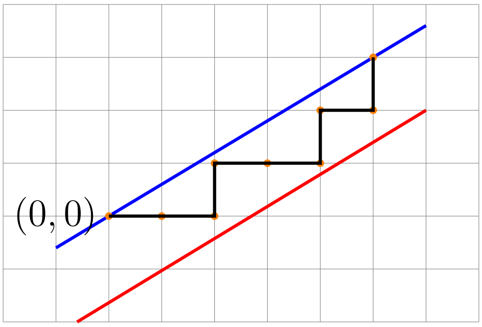


Fig. 3. The Reveillès standard segment $D[3, 5, 0]$, whose Freeman code is the primitive Christoffel word of slope $\frac{3}{5}$.

Table 1
Bijection between the sets of all the values of $f(\mu)$ and $\mathcal{L}(w)$, for $w = 00100101$.

μ	0	-1	-2	-3	-4	-5	-6	-7
$k = f(\mu)$	0	3	6	1	4	7	2	5
$w' \equiv_k w$	00100101	00101001	01001001	01001010	01010010	10010010	10010100	10100100

From μ to conjugates (and back). Fix $a \perp b$ and set $n = a+b$. Let w be the primitive Christoffel word of slope a/b read on the Cayley cycle, and let $\alpha \in \mathbb{Z}$ satisfy $\alpha a \equiv -1 \pmod{n}$ (so the 1-positions are $D_c(a, b) = \{(i\alpha) \bmod n : i = 1, \ldots, a\}$). For $\mu \in \{-a-b+1, \ldots, 0\}$ define $f(\mu)$ as the representative in $\{0, \ldots, n-1\}$ given by

$$f(\mu) = (\mu\alpha) \bmod n.$$

Then the Freeman code of $D[a, b, \mu]$ is the conjugate $w' \equiv_{f(\mu)} w$. Consequently, the set $\mathscr{R}(D[a, b, \cdot])$ of all such codes is in bijection with the conjugacy class $\mathscr{L}(w)$.

Remark 7. It is now straightforward to extend the definition of f to any $\mu \in \mathbb{Z}$. This also applies to the standard Réveillès segment.

Example 13. Let $w = 00100101$ be the primitive Christoffel word of slope $\frac{3}{5}$. Using Example 1, we have $\alpha = 5$ and $D_c(3, 5) = \{2, 5, 7\}$. Using the function f, all the conjugates of w are represented in Table 1.

Synchronization from a geometric point of view. The preceding discussion shows that, for fixed parameters (a, b) with $n = a + b$, varying the Réveillès parameter μ selects a conjugate of the corresponding Christoffel word. We now apply this identification to each row of the synchronized matrix $C(G, V)$.

Let $G = (g_1, \ldots, g_\ell)$ and $n = \sum_{i=1}^{\ell} g_i$. For each i, consider a standard Réveillès segment $D[g_i, n - g_i, \mu_i]$. Its Freeman code is a word of length n over the alphabet $\{0, 1\}$, where 0 denotes a right step and 1 denotes an up step. Changing the parameter μ_i changes the conjugate of the corresponding Christoffel word. Thus, in geometric terms, synchronization asks whether the parameters $\mu_1, \ldots, \mu_\ell$ can be chosen so that, for every position $j = 0, \ldots, n-1$, exactly one of the ℓ Freeman codes has the letter 1. Equivalently, among the ℓ digital segments, exactly one segment makes an up step at each time j.

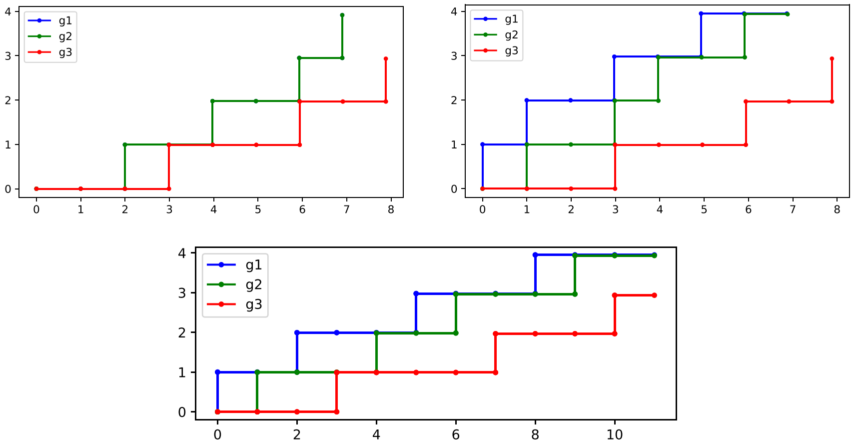


Fig. 4. 2D Graphical representation of the Christoffel-word matrix for $G = (4, 4, 3)$. Top, from left to right: the original 3 segments where g_1 and g_2 overlap, and the synchronized segments. Bottom, the synchronized segments after morphism.

Whenever $V = (v_1, \dots, v_\ell)$ is a seed for G, the matrix $C(G, V)$ records such a synchronized family of segments: row i records the up steps of $D[g_i, n - g_i, \mu_i]$, with each occurrence of 1 replaced by the row label i.

The following proposition records the precise correspondence between the algebraic seed V and the geometric parameters of the associated Réveillès segments. Although the verification is direct, the statement is useful because it identifies the rows of $C(G, V)$ with synchronized Freeman codes.

Proposition 5 (*Seed-to-geometry Correspondence*). *Let $G = (g_1, \dots, g_\ell)$, let $n = \sum_{i=1}^{\ell} g_i$, and let $V = (v_1, \dots, v_\ell)$ be a seed for G. For each i, let μ_i be the unique integer in $\{1 - n, \dots, 0\}$ satisfying $\mu_i \equiv -v_i \pmod{n}$. Then the ith row of $C(G, V)$ is exactly the Freeman code of the standard segment $D[g_i, n - g_i, \mu_i]$, with each letter 1 replaced by the row label i.*

Proof. By construction, the ith row of $C(G, V)$ records the decreases of the Cayley orbit starting at v_i. By the description of the positions of the letter 1 in conjugates, this orbit determines the same conjugate as the standard Réveillès segment whose parameter μ_i is the representative in $\{1 - n, \dots, 0\}$ congruent to $-v_i$ modulo n. Hence the Freeman code of $D[g_i, n - g_i, \mu_i]$ agrees with the ith row of $C(G, V)$, after replacing each letter 1 by the row label i. □

A note on the figures. In figures only, we may apply the morphism σ on the alphabet $\{0, 1, \dots, \ell\}$ defined by $\sigma(0) = 0$ and $\sigma(i) = i0$ for $i \geq 1$; this horizontally spreads vertical steps so all segments have the same horizontal span and one vertical per abscissa is visually obvious. The underlying synchronized words and codes are unchanged.

Example 14. For $G = (4, 4, 3)$ and the seed $V = (10, 5, 0)$ (so $n = 11$), Proposition 5 gives $\mu = (-10, -5, 0)$, i.e. $D[4, 7, -10]$, $D[4, 7, -5]$, $D[3, 8, 0]$. These are exactly the three rows of $C(G, V)$ in the worked Example 7. See Fig. 4.

Remark 8 (*Non-coprime Case*)**.** If $d = \gcd(a, b) > 1$, then

$$\mathrm{Ch}(a, b) = \mathrm{Ch}(a/d, b/d)^d.$$

Accordingly, the 1-positions form a union of d arithmetic progressions spaced by n/d, and the Réveillès description lifts componentwise.

Example 15. It is useful to revisit the Fraenkel family for three generators, which in our case is Example 10, with $G = (1, 2, 4)$. It is illustrated in Fig. 5.

For more examples, see B.

9. Conclusion and perspectives

We presented a compact algebraic framework for the synchronization (gap-free superposition) of Christoffel words of common length. Working on the Cayley cycle of $\mathbb{Z}/n\mathbb{Z}$ and using the two matrices $O(G, V)$ (orbital) and $C(G, V)$ (Christoffel–conjugate), we formulated synchronization as the one-decrease-per-column condition. A short *vertical invariant* shows that the column sums of $O(G, V)$ are constant if and only if this condition holds. Within this framework we obtained:

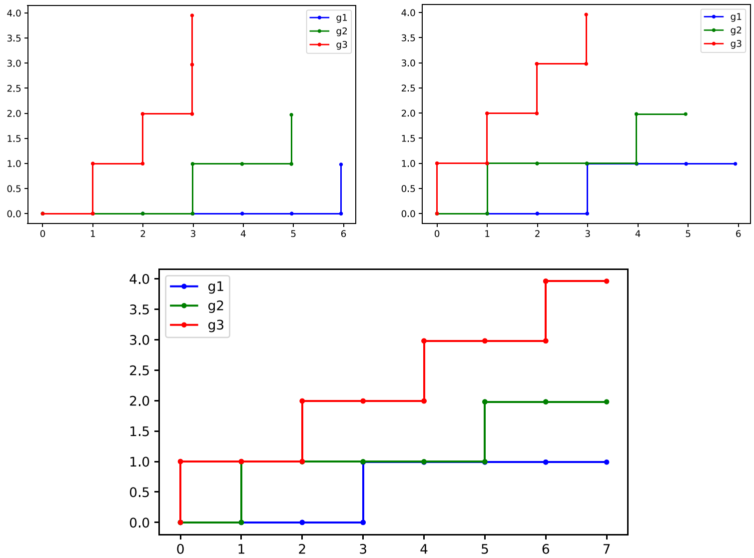


Fig. 5. The Fraenkel family for 3 generators – 2D graphical representation of the Christoffel-word matrix for $G = (1, 2, 4)$. Top-left: the original 3 segments; top-right: the synchronized segments; bottom: the synchronized segments after morphism.

- for $\ell = 2$, an explicit seed valid for all $G = (g_1, g_2)$ with $n = g_1 + g_2$;
- for $\ell = 3$, a complete solution when two generators coincide, and a uniform seed for the all-distinct family proportional to $(1, 2, 4)$.

The geometric viewpoint is given at the end of the paper: each synchronized row is the Freeman code of a standard 4-connected Réveillès segment, and the seed entry v_i determines its parameter via the unique representative $\mu_i \in \{1-n, \ldots, 0\}$ satisfying $\mu_i \equiv -v_i \pmod{n}$ (see Section 8).

Perspectives. Several directions appear promising:

1. **Three distinct generators.** Extend the $(1, 2, 4)$ family to broader classes of triples and clarify uniqueness. Characterize necessary/sufficient conditions on (g_1, g_2, g_3) for the existence of a seed.
2. **Beyond three words.** Generalize the seed criteria to $\ell > 3$ (motivated by Fraenkel's conjecture) and study the structure of synchronized words (distribution and regularity of row indices).
3. **Algorithmics.** Devise efficient procedures to decide synchronizability and to compute seeds, starting from the arithmetic descriptions of 1-positions in Section 4.
4. **Geometry.** Explore alternative geometric codings (e.g., different thickness or connectivity) and how they reflect algebraic constraints on $O(G, V)$ and $C(G, V)$.

Reproducibility. Python code reproducing all examples is available at:
https://github.com/Lama-Tarsissi/Synchronization-of-Christoffel-words.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Proof of Theorem 3

This appendix provides a complete modular proof of Theorem 3, including all parity cases and explicit bounds. It expands the argument outlined in the proof given in the main text.

Notation and preliminaries

Let $G = (g, g, g_3)$ with $n = 2g + g_3$ and $\gcd(g, g_3) = 1$. All congruences are taken modulo n, and all sets of positions are understood as subsets of $\{0, 1, \dots, n-1\}$.

Recall that for $\mathrm{Ch}(a, b)$, with $a \perp b$, whose Cayley orbit starts at label $p \in \mathbb{Z}/n\mathbb{Z}$, the set of positions of the letter 1 is

$$P(a, b, p) = \{(i\alpha + x) \bmod n : i = 1, \dots, a\},$$

where $\alpha a \equiv -1 \pmod n$ and $xa \equiv -p \pmod n$ (Section 4).

For a seed $V = (v_1, v_2, v_3)$ we define

$$A = P(g, n-g, v_1), \qquad B = P(g, n-g, v_2), \qquad C = P(g_3, n-g_3, v_3).$$

Synchronization is equivalent to the three sets A, B, C being pairwise disjoint.

Case 1: n odd

Assume that n is odd. By Proposition 1 we have

$$\gcd(n, g) = \gcd(n, g_3) = 1.$$

Let $\alpha, \beta \in (\mathbb{Z}/n\mathbb{Z})^\times$ satisfy

$$\alpha g \equiv -1 \pmod n, \qquad \beta g_3 \equiv -1 \pmod n.$$

We take the seed

$$V = (v_1, v_2, v_3) = \left(n-1, \tfrac{n-1}{2}, 0\right).$$

Let $x, y \in \mathbb{Z}/n\mathbb{Z}$ be defined by

$$xg \equiv -v_1 \equiv 1 \pmod n, \qquad yg \equiv -v_2 \equiv -\tfrac{n-1}{2} \pmod n.$$

Then

$$A = \{(i\alpha + x) \bmod n : 1 \le i \le g\},$$

$$B = \{(i\alpha + y) \bmod n : 1 \le i \le g\},$$

$$C = \{(j\beta) \bmod n : 1 \le j \le g_3\}.$$

Disjointness of A and B

Suppose $i\alpha + x \equiv j\alpha + y \pmod n$. Then $\alpha(i-j) \equiv y - x \pmod n$. Multiplying by g gives

$$-(i-j) \equiv (y-x)g \equiv \tfrac{n-1}{2} \pmod n.$$

Hence $j - i \equiv \frac{n-1}{2} \pmod n$. But $|j-i| \le g-1$ and $\frac{n-1}{2} \ge g$, since $n = 2g + g_3$ with $g_3 \ge 1$. This is impossible, so $A \cap B = \emptyset$.

Disjointness of A and C

Suppose $i\alpha + x \equiv j\beta \pmod n$. Multiply by gg_3:

$$(\alpha g)g_3\, i + (xg)g_3 \equiv (\beta g_3)g\, j \pmod n.$$

Using $\alpha g \equiv -1 \pmod n$, $xg \equiv 1 \pmod n$, and $\beta g_3 \equiv -1 \pmod n$, we obtain

$$-g_3 i + g_3 \equiv -gj \pmod n,$$

that is

$$gj \equiv g_3(i-1) \pmod n.$$

Reducing modulo g_3 yields $gj \equiv 0 \pmod{g_3}$. Since $\gcd(g, g_3) = 1$, this implies $j \equiv 0 \pmod{g_3}$, hence $j = g_3$.

Substituting back gives $g \equiv i - 1 \pmod n$, which forces $i = g + 1$, impossible. Thus $A \cap C = \emptyset$.

Disjointness of B and C

Suppose $i\alpha + y \equiv j\beta \pmod n$. Multiplying by gg_3 yields

$$-g_3 i - \tfrac{n-1}{2} g_3 \equiv -gj \pmod n.$$

Hence

$$gj \equiv g_3\Big(i + \tfrac{n-1}{2}\Big) \pmod n.$$

Reducing modulo g_3 again implies $j \equiv 0 \pmod{g_3}$, so $j = g_3$. Then

$$g \equiv i + \tfrac{n-1}{2} \pmod n,$$

Since $1 \le i \le g$ and

$$\frac{n-1}{2} = g + \frac{g_3 - 1}{2},$$

the integer $i + \frac{n-1}{2}$ lies in the interval (g, n). Hence the congruence

$$g \equiv i + \frac{n-1}{2} \pmod n$$

would force the equality $g = i + \frac{n-1}{2}$, which is impossible. Thus $B \cap C = \emptyset$.

We conclude that A, B, C are pairwise disjoint when n is odd.

Case 2: n even

Assume now that n is even. By Proposition 1 we have

$$\gcd(n, g) = 1, \qquad \gcd(n, g_3) = 2.$$

Write $n = 2m$. Choose $\alpha \in (\mathbb{Z}/n\mathbb{Z})^\times$ with $\alpha g \equiv -1 \pmod n$ and $\beta \in \mathbb{Z}/n\mathbb{Z}$ with $\beta g_3 \equiv -2 \pmod n$.

We take

$$V = (v_1, v_2, v_3) = \big(n-1, \tfrac{n-2}{2}, 0\big),$$

and define x, y by

$$xg \equiv 1 \pmod n, \qquad yg \equiv 1 + m \pmod n.$$

Then A and B are defined as before, while

$$C_0 = \{(j\beta) \bmod n : 1 \le j \le g_3/2\}, \qquad C_1 = C_0 + m, \qquad C = C_0 \sqcup C_1.$$

Here g_3 is even, so $g_3/2$ is an integer.

Disjointness of A and B

Suppose $i\alpha + x \equiv j\alpha + y \pmod n$. Multiplying by g gives

$$-(i-j) \equiv (y-x)g \equiv m \pmod n,$$

hence $j - i \equiv m \pmod n$. But $|j - i| \le g - 1 < m$, so this is impossible. Thus $A \cap B = \emptyset$.

Disjointness from C

It remains to show that neither A nor B meets C_0 or C_1. Let u denote either x or y, and suppose

$$i\alpha + u \equiv j\beta + rm \pmod n, \qquad r \in \{0, 1\}, \quad 1 \le i \le g, \quad 1 \le j \le g_3/2.$$

Multiplying by gg_3 and using

$$\alpha g \equiv -1 \pmod n, \qquad \beta g_3 \equiv -2 \pmod n, \qquad mg_3 = n(g_3/2) \equiv 0 \pmod n,$$

we obtain

$$-g_3 i + g_3(ug) \equiv -2gj \pmod n.$$

Since $ug \equiv 1 \pmod m$ for both $u = x$ and $u = y$, this congruence reduces, after division by 2 modulo $m = n/2$, to

$$gj \equiv \frac{g_3}{2}(i-1) \pmod m.$$

Write $h = g_3/2$. Since $m = g + h$, we have $h \equiv -g \pmod m$, and because $\gcd(g, m) = 1$, cancelling g gives

$$j \equiv -(i-1) \pmod m.$$

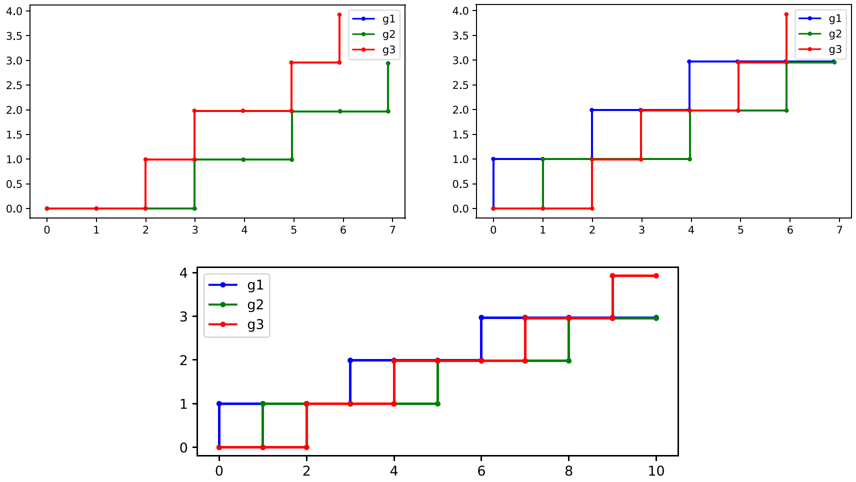

Fig. B.6. 2D Graphical representation of the Christoffel-word matrix for $G = (3, 3, 4)$. Top-left, the original 3 segments where g_1 and g_2 overlap; top-right, the synchronized segments; bottom, the synchronized segments after morphism.

But $1 \leq j \leq h$ and $0 \leq i - 1 \leq g - 1$, so

$$1 \leq j + i - 1 \leq h + g - 1 = m - 1.$$

Hence $j + i - 1$ cannot be congruent to 0 modulo m, a contradiction. Therefore

$$A \cap C_0 = A \cap C_1 = B \cap C_0 = B \cap C_1 = \emptyset.$$

Thus A, B, C are pairwise disjoint when n is even.

Conclusion

In both parity cases, the sets A, B, C are pairwise disjoint and satisfy

$$|A| + |B| + |C| = g + g + g_3 = n.$$

Hence

$$A \sqcup B \sqcup C = \{0, 1, \ldots, n - 1\}.$$

Equivalently, each column of $C(G, V)$ contains exactly one decrease. Therefore V is a synchronizing seed for $G = (g, g, g_3)$, with $v_3 = 0$ attached to the distinct generator.

Appendix B. Other Reveillès examples

Example 16. Looking at Example 8, we note that:
$D[3, 7, -9] = C(G, V)[0]$, $D[3, 7, -4] = C(G, V)[1]$, and
$D[4, 6, 0] = C(G, V)[2]$.
The synchronized word formed is $w = 1231321323$, see Fig. B.6.

Example 17. Looking at Example 9 ($G = (3, 3, 6)$), we get Fig. B.7.

Example 18. See Fig. B.8 for the example $G = (3, 3, 3)$, having as a seed vector $V = (6, 3, 0)$.

Data availability

No data was used for the research described in the article.

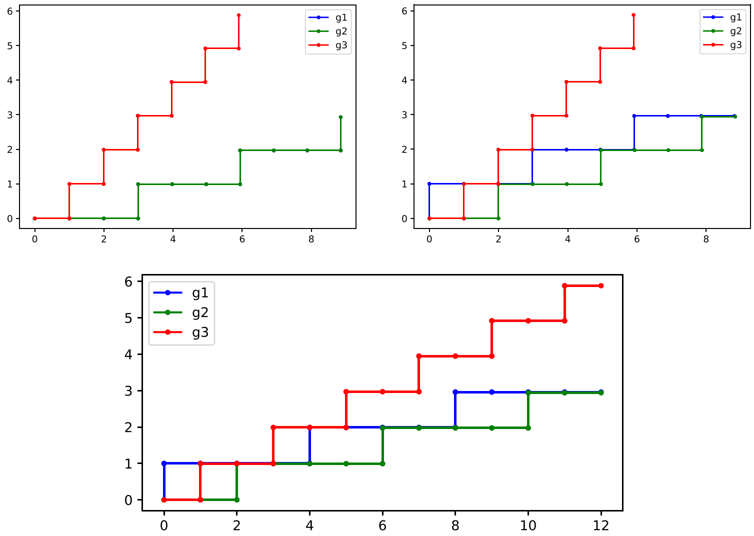

Fig. B.7. 2D Graphical representation of the Christoffel-word matrix for $G = (3, 3, 6)$. Top-left, the original 3 segments where g_1 and g_2 overlap; top-right, the synchronized segments; bottom, the synchronized segments after morphism.

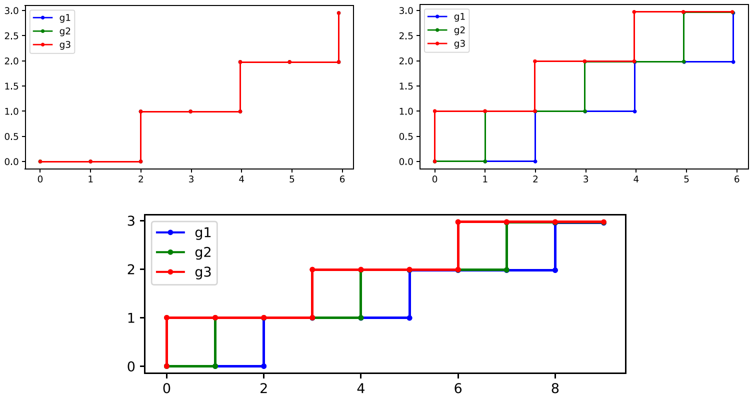

Fig. B.8. 2D Graphical representation of the Christoffel-word matrix for $G = (3, 3, 3)$. Top-left, the original 3 segments where g_1, g_2 and g_3 overlap; top-right, the synchronized segments; bottom, the synchronized segments after morphism.

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